Enhancing Efficiency of Mixing in Chaotic Flows

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Abstract

We propose a mechanism by which the efficiency of mixing in chaotic flows can be enhanced. Our mechanism consists of introducing small changes in the system parameters in regions of phase space where the local Lyapunov exponent falls substantially below its average value. We have applied our mechanism to several typical chaotic maps and flows including a system of chemical reactions. We find that our method is quite efficient as it gives a substantial enhancement of the rate of mixing with small changes in system parameters, without disturbing the attractor significantly.

PACS No. 05.45.+b

The importance of mixing has long been recognised in the context of physical systems [1]. Examples of mixing processes can be found in the context of combustion processes [2], fluid flows [3, 4, 5], viscous liquids [6], chemical reactions [7, 8], heat transfer processes [9] etc. Again, efficient mixing has desirable consequences in many practical contexts. A well mixed fuel–air mixture can lead to greater efficiency of the combustion process. Similarly, if the reactants of a chemical reaction are mixed better, this can lead to better yields of the resultants. Heat transport in convective processes can be enhanced by improved mixing. Many of these mixing processes can be modelled by chaotic flows [1]. Hence, if a mechanism can be found for the enhancement of the rate of mixing in chaotic flows, it can prove to be very useful in a variety of contexts. We propose such a mechanism in this paper.

Mixing is a consequence of the stretching and folding of chaotic flows. A system which has exponential stretching, as in a chaotic flow, mixes efficiently [1]. For a chaotic flow, the average rate of stretching can be characterized by the Lyapunov exponent. However, the rate of stretching is not uniform over the chaotic attractor. Thus the local Lyapunov exponent (LLE), a measure of the local rate of stretching, is different in different regions of the attractor [10]. We exploit the nonuniform nature of the spatial distribution of the local Lyapunov exponents to construct a mechanism that can enhance the rate of chaotic mixing. Briefly, we enhance the average rate of stretching by introducing a small parameter perturbation which enhances the local Lyapunov exponent whenever the system trajectory visits a region where the local Lyapunov exponents take values much smaller than their average value. We find that this procedure works quite efficiently as small perturbations in parameter made for small times compared to the total time of evolution can lead to substantial enhancement of the Lyapunov exponent and thus the mixing efficiency.

Let us consider an autonomous nonlinear dynamical system of dimension n, evolving via the equations $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mu)$, where the set of parameters μ takes values such that the trajectory shows chaotic behaviour. Let $\mathbf{w}(\mathbf{x}, t)$ be the tangent vector to the trajectory at the point $\mathbf{x}$ and time t. The evolution of $\mathbf{w}$ is given by $\dot{\mathbf{w}} = (\mathbf{x} \circ \nabla)\mathbf{F}$. The Lyapunov exponent of the system is defined by

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{||\mathbf{w}(\mathbf{x}, t)||}{||\mathbf{w}(\mathbf{x}(0), 0)||}. \tag{1}$$

where $\mathbf{x}(0)$ is the value of $\mathbf{x}$ at $t = 0$. We now define the local Lyapunov exponent $\lambda(\mathbf{x})$ as

$$\lambda(\mathbf{x}) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \ln \frac{||\mathbf{w}(\mathbf{x}(t + \Delta t), t + \Delta t)||}{||\mathbf{w}(\mathbf{x}(t), t)||} \tag{2}$$

Clearly $\lambda(\mathbf{x})$ represents the local rate of stretching at the point $\mathbf{x}$. This is, in general, not uniform over the attractor. We also note that the Lyapunov exponent λ (Eq.(1)) is the average value of the local Lyapunov exponents for a long orbit.

We set up a control procedure to enhance the mixing efficiency utilising the distribution of the local Lyapunov exponents. The control procedure operates in regions where the LLE-s fall substantially below the average value λ. If, at any time, the local Lyapunov exponent of the system falls below its average value to the point where

$$\lambda(\mathbf{x}) < (\lambda - \gamma\sigma_\lambda) \tag{3}$$

where σ_λ is the standard deviation of the distribution of LLE and γ is some chosen factor, the control is activated so that the parameter μ is changed to $\mu + s\, d\mu$. Here $d\mu$ is a small increment and s takes values $+1$ or -1 depending on which choice enhances the LLE. The system is allowed to evolve with the new value of the parameter as long as the condition (3) is satisfied. Thereafter the parameter is reset to its original value.

To decide the sign s, we write equation for $\mathbf{w}$ in matrix notation in the form

$$\dot{W}^T = W^T M^T, \;\; \dot{W} = MW; \tag{4}$$

where W^T is a row vector and the matrix M^T is given by $M^T = \nabla\, \mathbf{F}$. The equation for the norm of W can be written as

$$||\dot{W}||^2 \;\; = \;\; W^T(M^T + M)W \tag{5}$$

Thus the rate of change in the norm of W due to change in the parameter is given by

$$\begin{aligned} \Delta||\dot{W}||^2 &= ||W(\mu \dot{+} d\mu)||^2 - ||W(\dot{\mu})||^2 \\ &\simeq W^T(M_\mu^T + M_\mu)W\, d\mu \end{aligned} \tag{6}$$

where the last step is obtained by expanding to lowest order in $d\mu$ and $M_\mu = \partial M/\partial\mu$. Clearly, for the local rate of stretching to increase, $\Delta||W||^2$ must be positive . Thus the sign s is determined to ensure that $\Delta||W||^2$ is positive.

It must be noted that Eq.(6) is written in the lowest order in $d\mu$. Actually, the effect of the perturbation is nonlinear since when the parameter changes the entire trajectory of the system changes. Hence the effect on the LLE can be quite different from that given by Eq.(6) due to the effect of the higher nonlinear terms. In many cases the enhancement in the Lyapunov exponent turns out to be substantially higher than that expected in the linear approximation.

We now illustrate our procedure using some typical flows. We first consider the Lorenz system [11] with parameters $\sigma = 10.0$, $r = 30.0$ and $b = 8/3$. We

choose r as the control parameter. The perturbation is switched on when the condition (3) is satisfied. The sign of the perturbation dr is obtained using Eq.(6) and the sign is decided by $W_x W_y dr > 0$. For a change of parameter $dr = 1.0$, $\gamma = 0.5$, the Lyapunov exponent of the system is enhanced from $\lambda = 0.950$ for the uncontrolled case to $\lambda = 1.440$ (See Table I).

This enhancement in the Lyapunov exponent is not confined to the parameter values above. The plot of the Lyapunov exponent of the system as a function of r for both the uncontrolled and the controlled case is shown in Fig. 1. It is clear from the figure that there is a substantial enhancement of the Lyapunov exponent over the entire range plotted in the figure . This enhancement has been effected by causing a change in the local Lyapunov exponents of the system via parameter change. To show this, we plot the distribution of the LLE for $r = 30.0$ for both the uncontrolled and the controlled cases in Fig.2. It is clear that the distribution of local Lyapunov exponents of the system has changed in a manner in which the average exponent is significantly enhanced.

In order to show that the increase in the Lyapunov exponent translates into an enhancement of mixing efficiency, we operated the control procedure on a large number of initial conditions in a small region of phase space. We cover the attractor with a grid of cubic boxes. One of the box is chosen randomly. We take a large number of initial conditions in this box. Each initial condition is evolved according to our control algorithm i.e. the control is operative for a given trajectory (corresponding to a given initial condition) whenever the local Lyapunov exponent of the trajectory satisfies condition (3). The initial conditions are also evolved separately without the control. The initial conditions initially in one box spread over several boxes with time. A comparison of the number of occupied boxes, i.e. the boxes which have at least one initial condition, as a function of time for the uncontrolled and the controlled systems gives us an idea of the relative rates of mixing of the two systems. Fig.3 plots the number of occupied boxes as a function of time for both the uncontrolled and the controlled systems with a grid of 10^3 boxes and 10^5 initial conditions.The parameter values are $\sigma = 10.0, r = 30.0, b = 2.6666, dr = 1.0$ and $\gamma = 0.25$ [12]. It is clear from the figure that the controlled system mixes at a faster rate than the uncontrolled one. The results are unchanged for any randomly chosen initial box. This demonstrates that the control procedure has successfully enhanced the mixing efficiency of the system.

In order to demonstrate the efficacy of our procedure for a system of the type that constitutes our motivation, we apply our mixing algorithm to the Williamowski-Rossler attractor, which models a system of chemical reactions [7, 8]. The Williamowski-Rossler system evolves via the system of equations

$$\dot{x} = k_1 x - k_{-1} x^2 - k_2 xy + k_{-2} y^2 - k_4 xz + k_{-4}$$

$$
\begin{aligned}
\dot{y} &= k_2xy - k_{-2}y^2 - k_3y + k_{-3} \\
\dot{z} &= -k_4xz + k_{-4} + k_5z - k_{-5}z^2
\end{aligned} \tag{7}
$$

The system is allowed to evolve at the parameter values $k_1 = 30.0, k_2 = 1.0, k_3 = 10.0, k_4 = 1.0, k_5 = 16.5, k_{-1} = 0.25, k_{-2} = 1.0 \times 10^{-4}, k_{-3} = 1.0 \times 10^{-3} = k_{-4}, k_{-5} = 0.5$. Control is effected via a change in parameter k_1 whenever the condition (3) is satisfied. As seen from Table I this results in a large enhancement of the Lyapunov exponent from the uncontrolled value $\lambda = 0.559$ to the value $\lambda = 0.804$ after the application of the control. We have also verified that the rate of mixing is enhanced due to the control by evolving a large number of initial conditions in a small region of phase space.

The mixing procedure discussed above can be easily modified to apply to the case of maps. In this case the condition (6) gets modified to the form

$$
\Delta||W||^2 \;=\; W^T(MM_\mu^T + M_\mu M^T)W\,d\mu \tag{8}
$$

For control to enhance mixing efficiency, the parameter change $d\mu$ must be such that $\Delta||W||^2$ is positive. If this procedure is applied to the Henon map [13] at parameter values $a = 1.2, b = 0.3$ with $da = 0.1$ as the parameter change, we again find an enhancement of the Lyapunov exponent from the uncontrolled value $\lambda = 0.306$ to the controlled value $\lambda = 0.328$ (See Table I).

The increase in the rate of spread of initial conditions in phase space of the controlled Henon system as compared to the uncontrolled one is demonstrated in Fig. 4 ($a = 1.1, b = 0.3, \gamma = 0.5$ and $da = 0.2$).Figs 4(a) and 4(b) show the uncontrolled and controlled systems respectively after 10 iterations. At this stage itself the controlled system has spread out more than the uncontrolled situations. This difference can be even more clearly seen in Figs 4(c) and 4(d) which show the uncontrolled and controlled systems after 1500 iterations. The Lyapunov exponent has changed from $\lambda = 0.177$(uncontrolled) to $\lambda = 0.258$ (controlled).

Thus our control procedure works for all the maps and flows tested including the case of the chemical reaction system and the Lyapunov exponent is substantially enhanced in most cases. However, it is important to ensure that the control does not disturb the attractor unduly. Visual comparisons of the appearance of the controlled and uncontrolled attractors reveal no significant differences between them. A more quantitative comparison can be made by comparing the fractal (box-counting) dimensions of the two. In the case of the Lorenz system, for the parameter values listed in Table I, the fractal dimension of the uncontrolled attractor was $D_0 = 2.052$ whereas that of the controlled attractor was $D_0 = 2.056$. In the case of the Henon attractor, again for the values of Table I, the fractal dimension changed insignificantly from $D_0 = 1.206$

to $D_0 = 1.212$. For the Williamowski-Rossler attractor, the fractal dimension remained practically unchanged at 2.07. Thus the control procedure does not appear to disturb the attractor unduly at the present values of parameter change.

The Lyapunov exponent referred to in the entire discussion above is the largest Lyapunov exponent of the system. A possible reason for the insignificant increase in dimension seen despite the application of control can be found in the values of the other Lyapunov exponents of the system. For the Lorenz attractor, for the parameter values $\sigma = 10.0, r = 30.0, b = 2.6666, dr = 1.0$ and $\gamma = 0.5$, the complete set of Lyapunov exponents of the uncontrolled system is given by $(0.950, 0.000, -14.617)$, and that of the controlled system is given by $(1.440, -0.420, -14.686)$. Thus the largest Lyapunov exponent of the system, which is a measure of the rate of stretching, has increased whereas the other Lyapunov exponents of the system, have become more negative signifying an increase in the rate of contraction. The controlled system no longer has a zero Lyapunov exponent as the trajectory is no longer smooth. These observations are also true of the other systems studied. The control procedure tends to push the trajectory in the basin of attraction of the uncontrolled attractor, but the increased rate of contraction pushes it back to the original attractor. Both these factors work to our advantage. As mentioned earlier, the increase in the rate of stretching tends to mix the system better. The increase in the rate of contraction has the advantage that it tends to stabilise the attractor, so that the attractor is not unduly disturbed by the perturbation. This is the origin of the insignificant change in the fractal dimension of the controlled and uncontrolled attractors. However, for large changes in parameter the difference between the fractal dimensions of the controlled and uncontrolled attractor does increase. Again the stability of the uncontrolled attractor plays an important role in this. Although the Lorenz attractor remains stable for large changes in parameter, and does not show a large increase in dimension, the difference increases substantially for the Henon attractor.

The control procedure outlined above leads to an enhanced rate of mixing for most parameter settings. However, we did find a few cases where it did not work well, e.g . in the neighborhood of the parameter values $r = 138.0$ and $r = 160.0$ for the Lorenz attractor. This happened for parameter values where there was a wide periodic window nearby. In such cases the control tends to push the trajectory in the neighbourhood of a periodic orbit. As a consequence the trajectory appears to show intermittent behaviour and the Lyapunov exponent does not increase and sometimes even decreases. This problem can be easily taken care of by changing the magnitude of the parameter change and/or the factor γ.

Our control procedure works quite efficiently as it produces a substantial enhancement of the Lyapunov exponent for quite small changes in the parameters. This is due to the fact that our control procedure works by switching between three types of chaotic flows, those characteristic of parameter values μ, $\mu + d\mu$ and $\mu - d\mu$. This switching introduces an extra time dependence in the problem and is the origin of the efficiency of the procedure.

To summarise, we have introduced an efficient mixing mechanism that produces a substantial increase in the rate of mixing for small changes in parameters. The chaotic attractor is not disturbed unduly. The success of the mechanism has been demonstrated for several chaotic flows and maps. We hope that this mechanism will prove to be useful in enhancing the rate of mixing in a variety of practical contexts.

We thank the Department of Science and Technology(India) for financial assistance. We thank IUCAA (Pune, India) and IMSc (Madras, India) for the use of their computing systems.

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Table Caption

We list the uncontrolled (Free) and controlled (Cont.) values of the Lyapunov exponent and of the fractal dimensions for several maps and flows. The values of the parameters of the systems analysed are listed in the text. The column Fract. refers to the fraction of time for which the system is controlled and $d\mu$ is the parameter change.

System	γ	$d\mu$	Fract.	Lyapunov exp.		Dimension	
				Free	Cont.	Free	Cont.
Lorenz	0.5	$dr = 1.0$	0.344	0.951	1.440	2.052	2.056
	0.25	$dr = 1.0$	0.447	0.951	1.362	2.052	2.061
Williamowski-Rossler	1.0	$dk_1 = 1.5$	0.047	0.559	0.804	2.069	2.068
Henon	0.5	$da = 0.1$	0.263	0.306	0.328	1.206	1.212

Figure Caption

Fig.1 The plot of the Lyapunov exponent of the Lorenz attractor for the parameters $\sigma = 10.0, b = 2.6666$ and r from $r = 28.0$ to $r = 80.0$. The lower curve corresponds to the Lyapunov exponent for the uncontrolled system, the upper to the controlled system with $\gamma = 0.5$ and $dr = 1.0$

Fig.2 A histogram of the distribution of the local Lyapunov exponents of the uncontrolled and controlled Lorenz systems for the parameter values $\sigma = 10.0, r = 30.0$, $b = 2.6666$,$\gamma = 0.5$ and $dr = 1.0$. The data has been binned into ten boxes. Each box is divided into two parts. The bar occupying the left half of the box shows the normalised frequency of occurance of the corresponding LLE of the uncontrolled system and the bar occupying the right half of the box (the bar with vertical lines) shows the same quantity for the controlled system.

Fig.3 The plot of the number of occupied boxes as a function of time for the uncontrolled (solid line) and controlled (dashed line) Lorenz systems. The parameter values are $\sigma = 10.0, b = 2.66666, r = 30.0, \gamma = 0.25$ and $dr = 1.0$.

Fig.4 We show the spread of 1000 points, initially in the same box of a 128 by 128 grid on the Henon attractor. Fig 4(a) and Fig. 4(b) show the uncontrolled and controlled Henon system after 10 iterates. Fig. 4(c) and Fig. 4(d) show the uncontrolled and controlled Henon system after 1500 iterates. The parameter values are $a = 1.1, b = 0.3$ and $\gamma = 0.5$ and $da = 0.2$.

Fig. 1 (NG & REA)

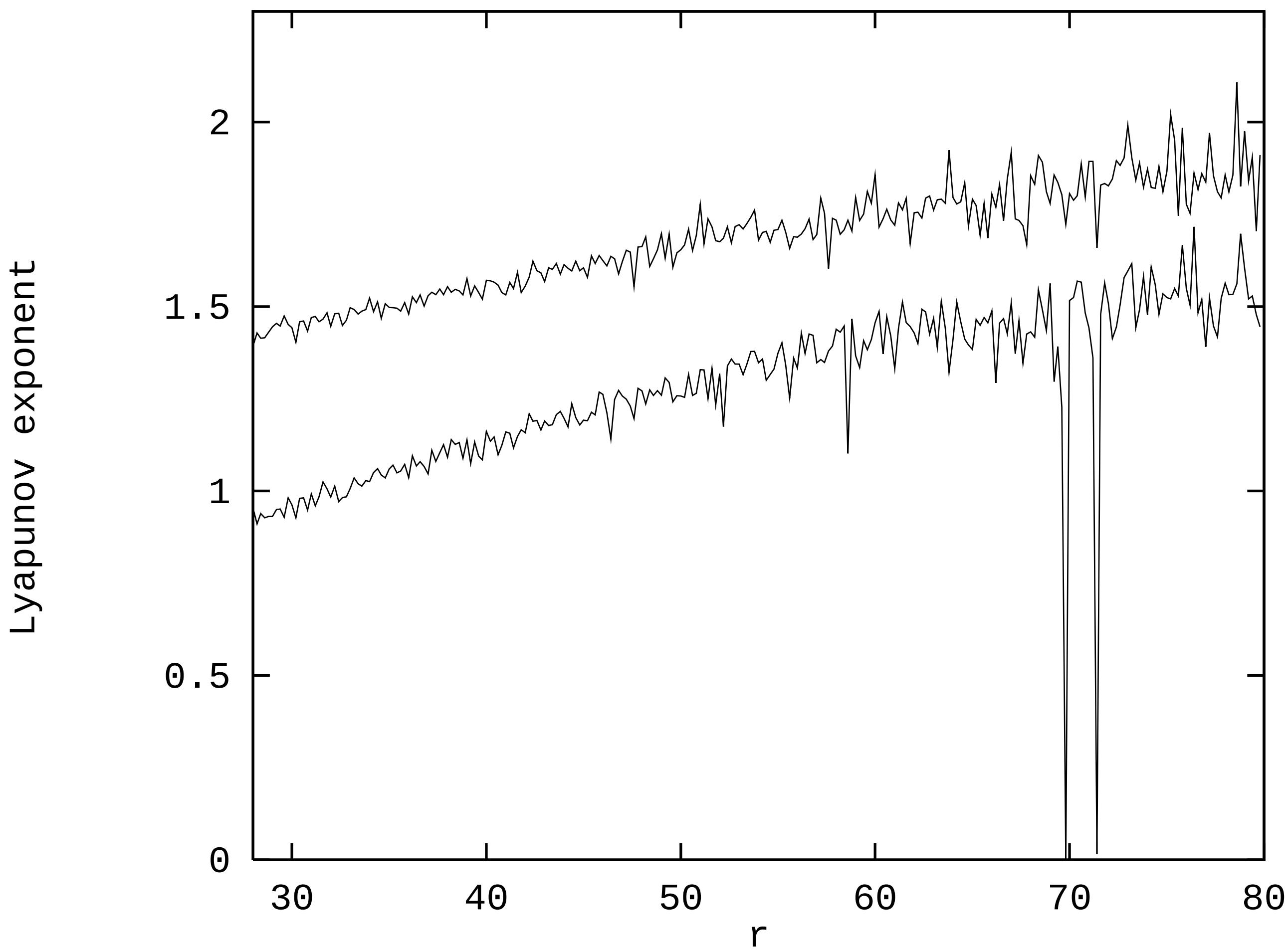
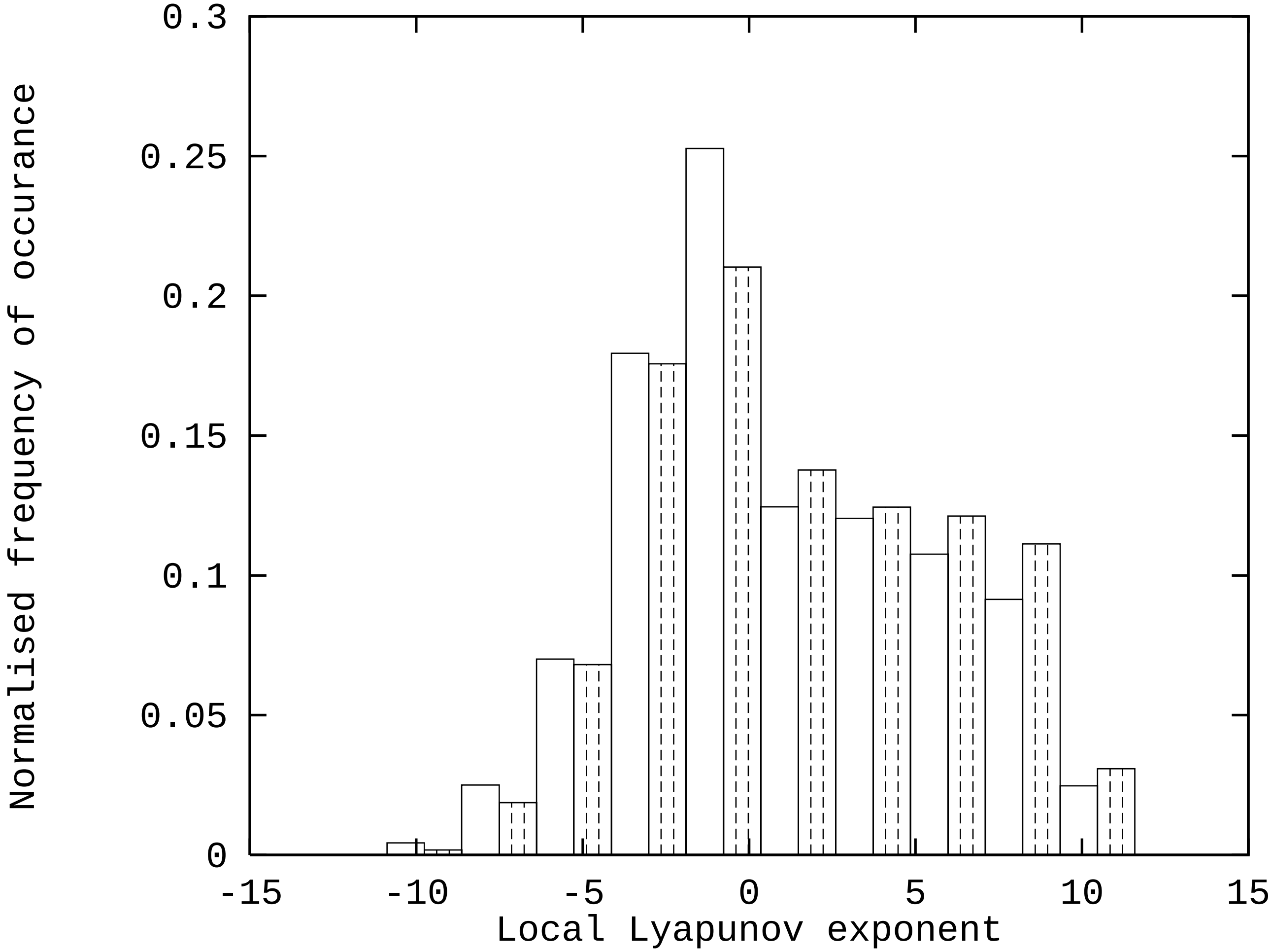
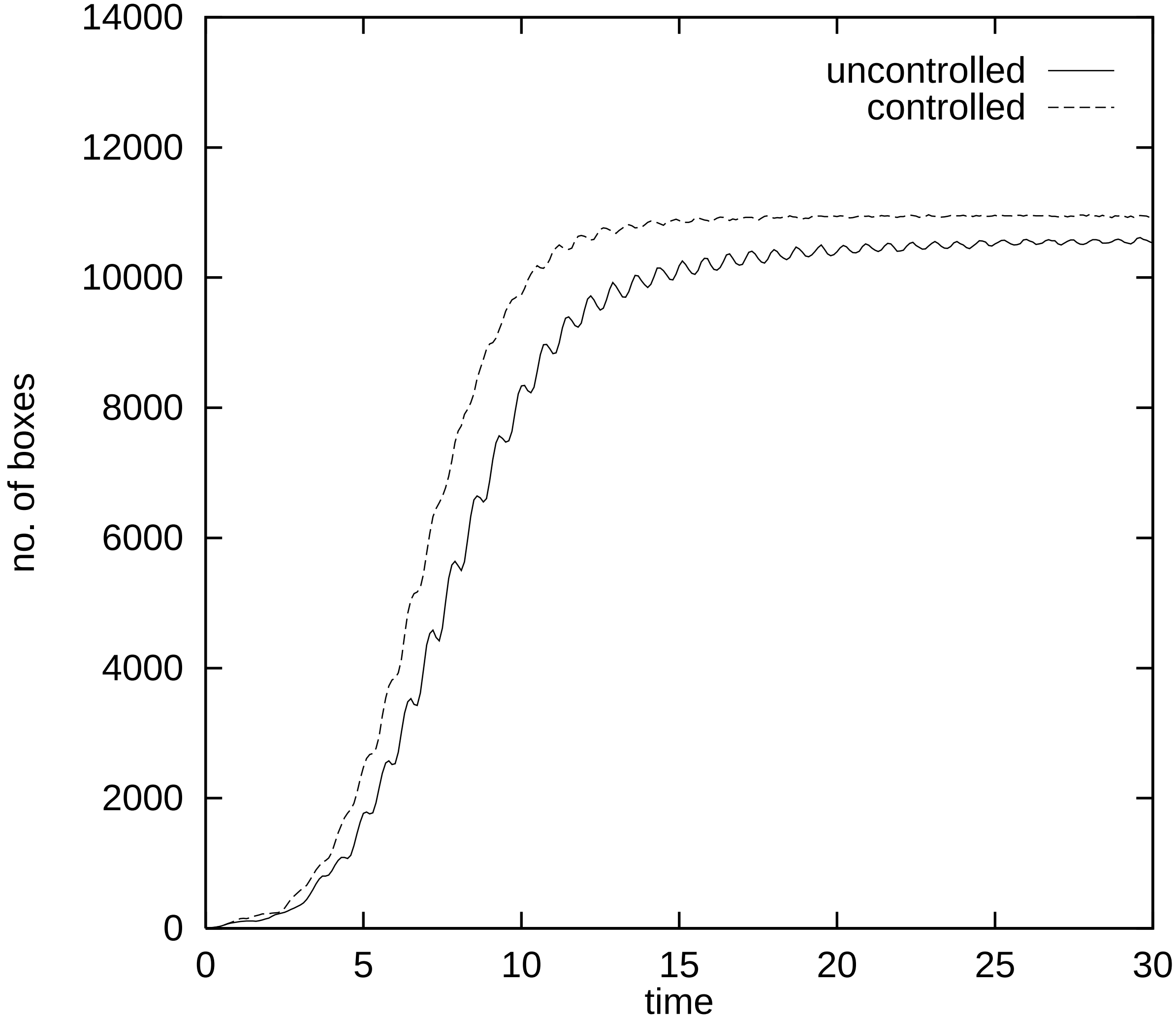


Fig. 2 (NG & REA)



0.3
0.25
0.2
0.15
0.1
0.05
0
-15
-10
-5
0
5
10
15
Local Lyapunov exponent
Normalised frequency of occurance

Fig. 3 .... (NG & REA)



14000
12000
10000
8000
6000
4000
2000
0
0
5
10
15
20
25
30
time
no. of boxes
uncontrolled
controlled

Fig. 4(a) ....   (NG & REA)

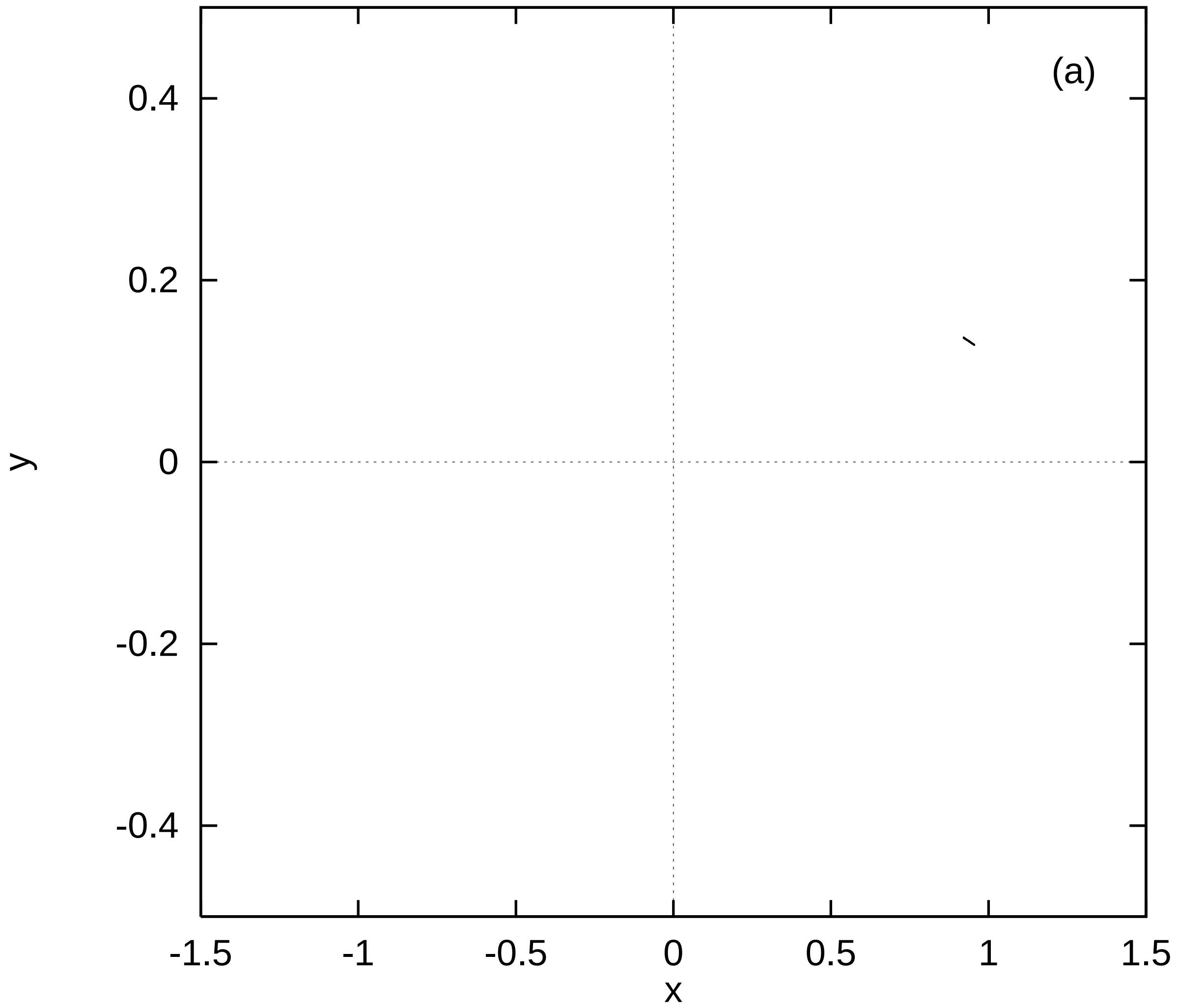


Fig. 4(b) .... (NG & REA)

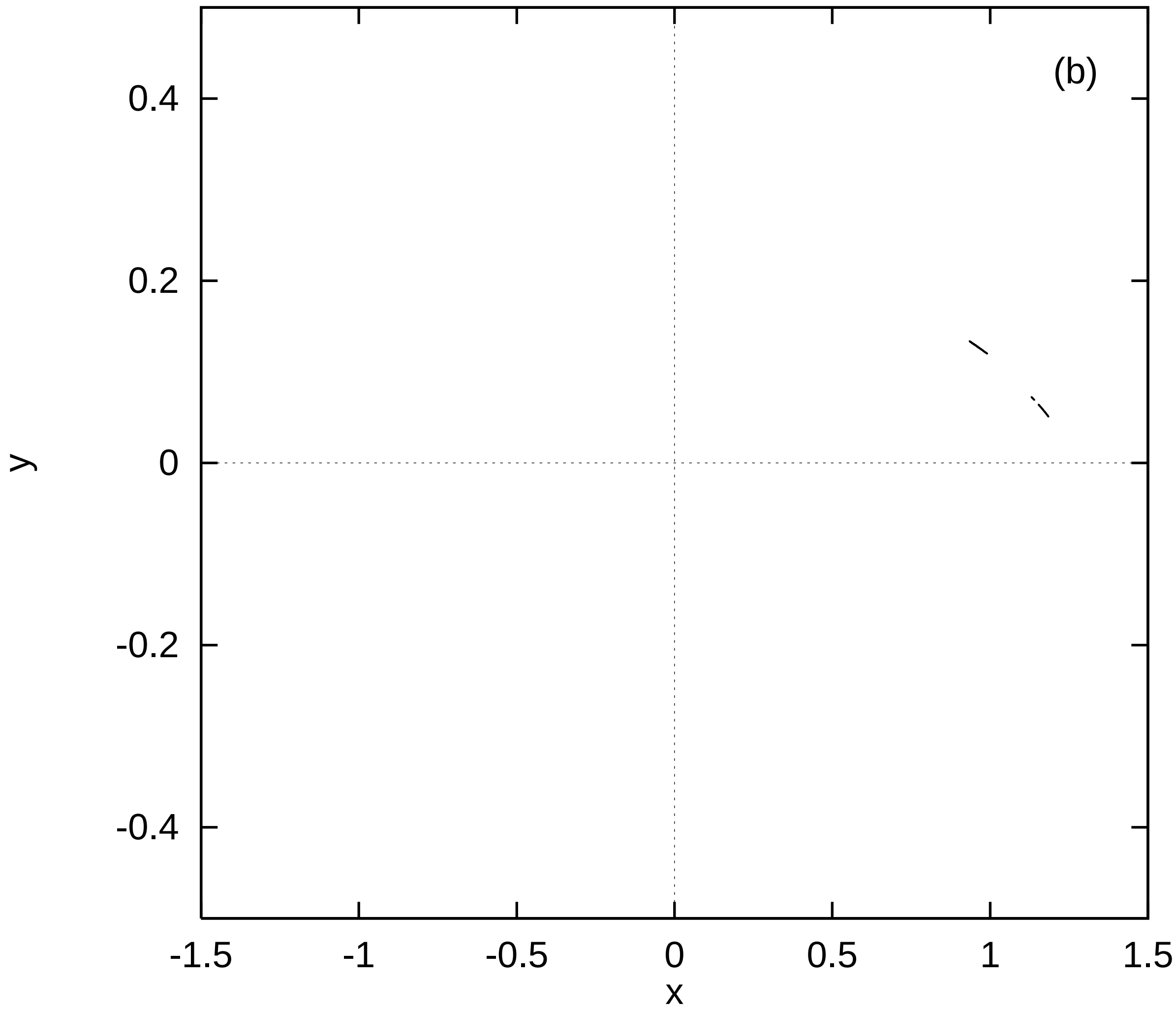


Fig. 4(c) .... (NG & REA)

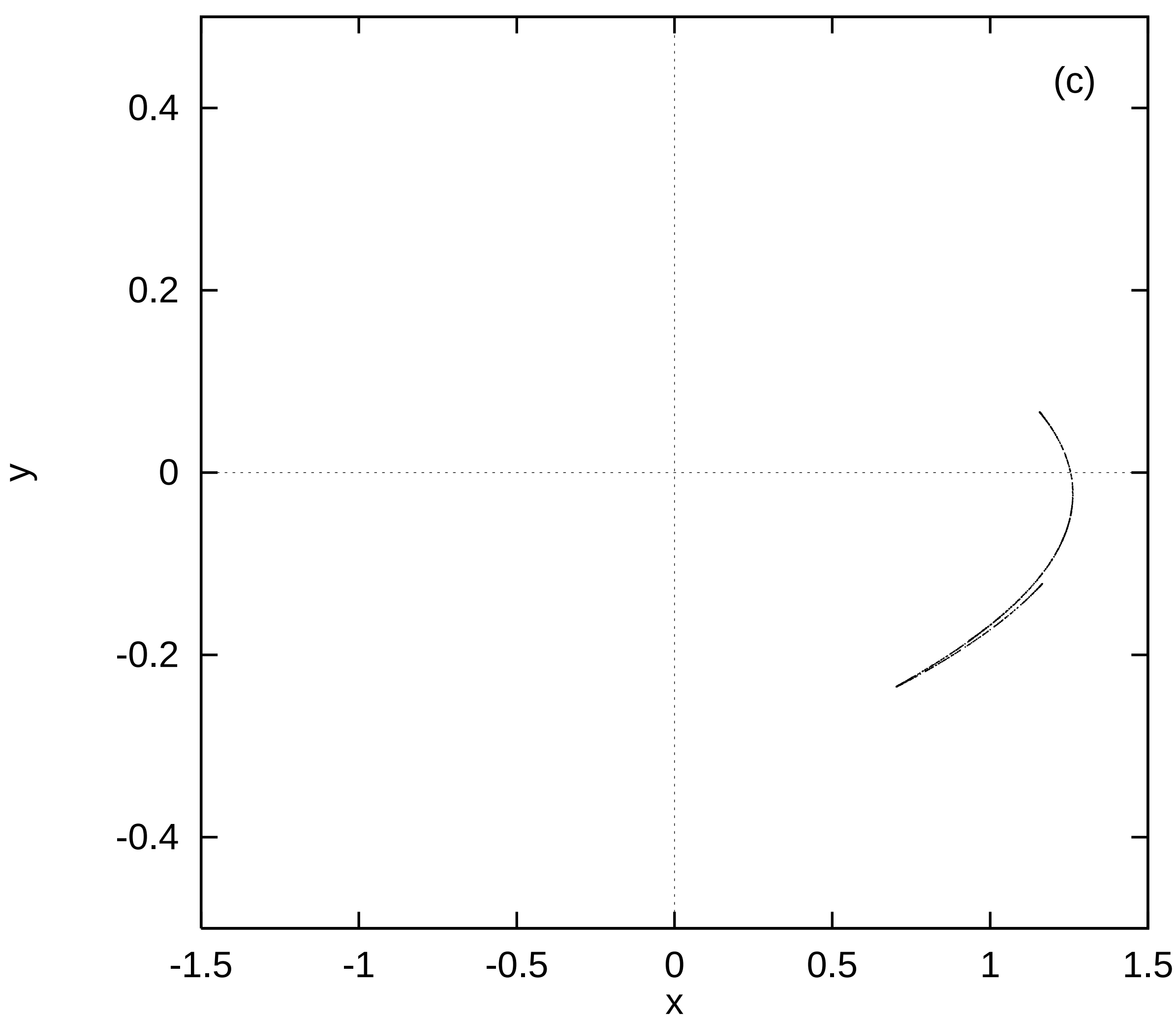


Fig. 4(d) ....   (NG & REA)